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The unsteady lift on a swept blade tip

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Highly swept blades are now commonly used in modern aeroengines, and in this paper we solve a model problem of relevance to the understanding and prediction of noise generation by the interaction between incident vortical disturbances and such a blade. In order to include the potentially significant effects of the blade-tip region, we consider a (semi-infinite span) quarter-plane aligned at a non-zero sweep angle to supersonic mean flow, with a single harmonic gust incident on the quarter-plane from upstream. The solution is completed using a novel application of the Wiener–Hopf technique, in which the usual factorisation is carried out with respect to two independent complex variables separately, and closed-form expressions for the practically important lift per unit span are derived for both the subsonic and the supersonic leading-edge regimes. The dependence of the unsteady response on the sweep angle and the gust wavenumbers is examined, and in particular we demonstrate that the magnitude of the effects of the tip region is significantly reduced by increasing the blade sweep or by considering gusts of higher frequency. It also becomes clear that the magnitude of the unsteady response can be either decreased or increased by sweeping the blade, in a way which proves highly dependent on the particular values of the flow parameters. In addition, we consider the two critical values of the gust trace speed along the leading edge which correspond to the sonic velocities in the two spanwise directions, and for which the chordwise oscillation of the unsteady lift distribution on an infinite-span blade vanishes. In these two cases, the lift per unit span (integrated over the infinite chord) is clearly singular, but we demonstrate that the effect of including the blade tip is to smooth out just one of these singularities, and replace it instead by a relatively large, but finite, value.

1. Introduction

The blades used in modern propellers and ducted fans are often highly swept, in order to improve the aerodynamic performance by reducing the leading-edge normal velocity near the transonically moving blade tip, and in addition this can have the added benefit of significantly reducing the noise generated (see for instance Hanson 1980), by introducing a radial dephasing of the effective sources along the blade span. The noise will typically be composed of a number of different components, but for ducted-fan systems and contra-rotating propellers the most significant noise-generation mechanism involves the interaction between some vortical disturbance, perhaps corresponding to the wake shed from an upstream row, and the rotating blades. In order to predict this noise one requires the unsteady response of the blade to the incident disturbance (i.e. the induced unsteady lift distribution on the blade surface), which can then be substituted into standard radiation integrals corresponding to a summation over effective sources along the blade span. As we have already mentioned, one effect of blade sweep is to reduce the noise generated by changing the radial *phase* of these effective sources; however, sweeping the blade will also change the source *strength*, and it is not clear *a priori* whether the sweep will reduce the magnitude of the unsteady response (thereby tending to reduce the noise still further), or increase it (thereby cancelling out either partially, or perhaps even completely, the noise benefits gained from the adjustment of the radial phasing). In this paper we shall therefore consider the effect of the sweep on the unsteady response, and it will indeed emerge that sweeping the blade can under certain circumstances lead to a marked increase in the effective source strength.

We consider a single harmonic velocity gust, as a simple component of some more complicated disturbance, which is convected by a uniform supersonic free stream and which is scattered by a thin blade aligned at non-zero sweep angle but zero angle of attack to the flow. One could further suppose that the blade has an infinite span (that is, it possesses an infinite extent in both spanwise directions), and a considerable body of work has been completed on such a system, for both subsonic and supersonic mean flows; see for instance Adamczyk (1974), Martinez & Widnall (1980) and Amiet (1976), and Landahl (1989) with particular reference to transonic mean flow. However, since modern blades typically possess a span of only a few chord lengths, and since a significant amount of the noise generation will often occur near the blade tip, the use of such infinite-span response theory seems inappropriate, and some account of the effects of the combination of the presence of the blade tip and the sweep needs to be made (we note here that Envia & Kerschen 1984 have studied the unsteady flow past a swept, finite-span airfoil in a duct). We therefore consider the interaction of a gust with a (semi-infinite span) quarter-plane in order to better model the tip effects, and consider only gusts with a moderate or high reduced frequency, so that the effects of the trailing edge can be neglected as a first approximation. The scattering of gusts and sound waves by unswept quarter-planes has attracted some interest in the past; Miles (1951) has considered an oscillating quarter-plane in supersonic mean flow, whilst Martinez & Widnall (1983) have developed an approximate solution for the case of subsonic flow, and Peake has studied the equivalent problem in transonic flow (Peake 1992) and the interaction between a steady jet and an unswept quarter-plane in supersonic flow (Peake 1993). The analysis presented here can therefore be thought of as an extension of much of this work to include the effects of sweep. Although fan and propeller blades do not have rectangular tips in practice, the use of a swept quarter-plane here seems an appropriate first step, which both captures the main features of the problem and which is amenable to analytical solution.

In §2 we present the formulation and solution of the problem, considering the two cases of the mean flow possessing either a supersonic component normal to the leading edge (i.e. relatively little sweep) or a subsonic component (i.e. a more significant amount of sweep). Here we employ a rather novel application of the standard Wiener–Hopf technique, in which we Fourier transform along both the spanwise and chordwise directions, and then make a multiplicative decomposition of the usual Wiener–Hopf kernel with respect to *both* integration variables in succession. Using this method we are able to derive an integral expression for the scattered field, but since it has not proved possible to manipulate this result into a closed-form expression, some simplification is required, and could be made by either considering the radiation far from the blade using the method of stationary phase, or by developing an expression for the chordwise-integrated unsteady lift (the *lift per unit span*) on the quarter-plane. However, the radiation in our simplified problem seems to be of little relevance to the actual radiation produced by a rotating blade, and we therefore proceed by deriving a closed-form expression for the lift per unit span; as well as being very closely related to the effective radial source strength along the blade required in noise prediction (see Peake 1992, Appendix B for full details), this provides an illuminating and convenient way of assessing the level of the unsteady response.

In §2 we also consider the two critical values of the component of the gust phase-front trace speed along the leading edge which correspond to the respective sonic velocities in the two spanwise directions. For *both* these two critical values, the chordwise oscillation associated with the unsteady lift on the infinite-span airfoil vanishes (i.e. infinite chordwise wavelength), so that the lift per unit span integrated over the infinite chord does not converge, and is singular. However, it will be shown that the lift per unit span on our quarter-plane is only singular for the critical value corresponding to the sonic velocity along the span in the direction towards the tip, and that it takes a relatively large, but finite, value for the other critical leading-edge trace speed. From this we are able to deduce that the disappearance of the chordwise oscillation, and consequent infinities in our expression for the lift per unit span, arise from the unrealistic infinite extent of our model blades in one or both spanwise directions. Further, it can be seen that the lift per unit span on a finite-span blade will typically possess its largest value when the gust wavenumbers and sweep angle combine in such a way as to yield a leading-edge trace speed equal to one of these two critical values.

The behaviour of the lift per unit span is assessed in §3 for various typical sets of parameter values, and we demonstrate in particular how the level of the effect of the presence of the tip on the unsteady lift *decreases* as the sweep angle is increased, and that how, under certain circumstances, the lift per unit span can actually be *increased* by sweeping the blade.

Finally, we again emphasize that the expressions for the lift per unit span presented in this paper have been determined by integrating the detailed unsteady lift distribution along the (supposed) infinite chord of either the infinite-span blade or the quarter-plane, leading to the divergence of our lift per unit span in the critical cases where the chordwise oscillation vanishes. An alternative (more accurate) approach, used by Landahl (1989), is to determine the lift distribution for the infinite-chord blade as above, and then to integrate this numerically over a realistic finite chord length. Given the highly complicated nature of the various Fourier integrals, which would also require numerical inversion, this second approach has not been attempted for the present problem. However, for the large reduced frequencies, Ω, considered, it can be shown that, provided we are not too close to the critical cases in which the chordwise oscillations disappear, the lift per unit span calculated in this paper is $O(\Omega^{-1})$ – see (2.16) – and that the discrepancy between our results and those which would be obtained using Landahl's method is $O(\Omega^{-\frac{3}{2}})$, due to the strong chordwise cancellation effects away from the leading edge. For most parameter values the appraoch adopted here will therefore be in reasonable agreement with Landahl's method, so long as Ω is moderately large. However, when the chordwise oscillations become very weak (or disappear completely), the lift per unit span calculated by integrating along an infinite chord will become very large (or infinite), and a meaningful answer can then only be obtained using Landahl's approach (of course, the infinities in our results indicate the parameter values for which the response of the real finite-chord blade is likely to be largest). A significant result of this paper is that the inclusion of a blade corner in the model reduces the number of critical sonic velocities for which the chordwise oscillations disappear from two to one. This then suggests that when considering a

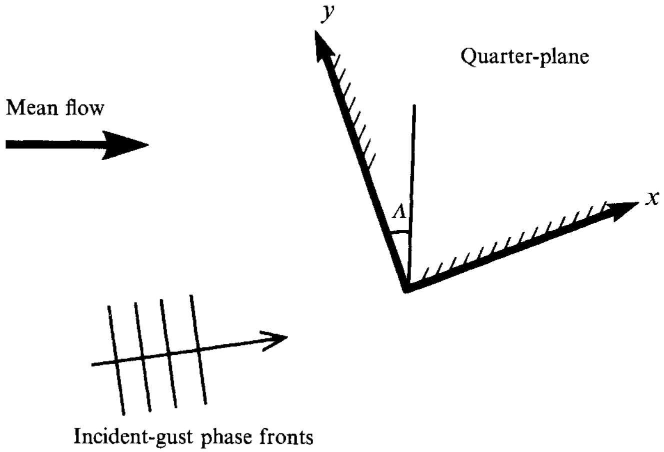

FIGURE 1. Plan view of the system (the z-axis points out of the plane).

model blade with a tip, the parameter range over which our approach of integrating along a supposed infinite chord gives meaningful results is considerably greater than that for a simple 'two-dimensional' analysis of an infinite-span blade.

2. Mathematical solution

2.1. *Formulation*

We consider a semi-infinite-span rigid blade of zero thickness with a chord of uniform length c, and choose the y-axis to be aligned along the blade leading edge and the x-axis along the side edge. There is a uniform supersonic mean flow of speed U in a direction making a positive angle Λ with the x-axis (so that Λ is the angle of sweep of the blade), and the mean-flow Mach number is M ($M > 1$). The system is shown in figure 1. In what follows, lengths are non-dimensionalized by c, time by c/U, velocities by U, densities by the uniform density of the fluid ρ_0 and pressures by $\rho_0 U^2$. We suppose that a convected harmonic velocity gust is incident on the blade from upstream, so that the normal-velocity upwash on the blade is of the form

$$V \exp(\mathrm{i}\Omega t - \mathrm{i}\alpha_1 x - \mathrm{i}\alpha_2 y)\mathbf{z} \ , \tag{2.1}$$

where Ω is the reduced frequency of the system ($\Omega \equiv \omega c/U$, with ω the dimensional gust frequency), α_1 and α_2 are the gust wavenumbers associated with the chordwise and spanwise directions, V is the gust magnitude and $\mathbf{z}$ is the unit vector perpendicular to the blade. The gust wavenumbers associated with the directions parallel to and perpendicular to the mean flow are Ω (since the convected gust must possess zero acoustic pressure) and k_p respectively, and expressions for $\alpha_{1,2}$ as functions of Ω, k_p and Λ can be derived via simple trigonometry. We suppose that $V \ll 1$, so that the scattered field generated by the interaction between the gust and the blade will be irrotational and will be governed by linear theory (see Goldstein 1976), with the velocity potential $\phi(x, y, z)\exp(\mathrm{i}\Omega t)$ satisfying the convected wave equation

$$(1 - M_1^2)\frac{\partial^2 \phi}{\partial x^2} + (1 - M_2^2)\ \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} + 2M_1 M_2 \frac{\partial \phi}{\partial x \partial y}$$
$$+2\mathrm{i}M\Omega\left(M_2\frac{\partial \phi}{\partial y} - M_1\frac{\partial \phi}{\partial x}\right) + M^2\Omega^2\phi = 0 \ , \tag{2.2}$$

where $M_1 = M\cos\Lambda$ and $M_2 = M\sin\Lambda$ are the Mach numbers corresponding to the components of the mean flow normal to and along the blade leading edge. In this paper we shall suppose that the side edge is always subsonic ($M_2 < 1$), but will allow the leading edge to be either subsonic ($M_1 < 1$) or supersonic ($M_1 > 1$). The effects of the trailing edge of the blade on the scattered field are likely to be small compared to those of the leading and side edges, particularly for moderate or large values of Ω, and in what follows we therefore neglect the presence of the trailing edge in our analysis, and thereby represent the semi-infinite blade by a quarter-plane lying in the first quadrant of the (x, y)-plane. The chord length c therefore only appears in the analysis when non-dimensionalizing the lengths and in the definition of the reduced frequency. The condition of zero normal velocity on the blade therefore becomes

$$\frac{\partial\phi}{\partial z}(x, y, 0) + V\exp(-\mathrm{i}\alpha_1 x - \mathrm{i}\alpha_2 y) = 0 \quad \text{for } x > 0, y > 0 \ , \tag{2.3}$$

and we shall also require that the acoustic pressure of the scattered field is continuous across the plane $z = 0$, apart from across the quarter-plane. We must ensure that the solution is causal, and to facilitate this we suppose that Ω possesses a small negative imaginary part. In addition, in order that the subsequent Fourier transform of (2.3) converges, we shall require that k_p also possesses a small negative imaginary part in such a way that both α_1 and α_2 lie in the lower half of the complex plane; these imaginary parts are all set to zero at the end of the analysis.

2.2. *Wiener–Hopf analysis and solution*

The solution will be completed using Fourier transforms, with

$$\Phi(k_1, k_2, z) \equiv \int_{-\infty}^{\infty} \phi(x, y, z)\exp(\mathrm{i}k_1 x + \mathrm{i}k_2 y)\mathrm{d}x\mathrm{d}y \ , \tag{2.4}$$

and by transforming the wave equation with respect to x and y it can be shown that

$$\Phi(k_1, k_2, z) = \tfrac{1}{2}\operatorname{sgn} z\,[\Phi(k_1, k_2, 0)]_-^+ \exp(-\mathrm{i}\gamma|z|) \ , \tag{2.5}$$

where $[\Phi(k_1, k_2, 0)]_-^+$ is the Fourier transform of the jump in $\phi(x, y, z)$ across $z = 0$ and where

$$\gamma^2(k_1, k_2) = M^2\Omega^2 - (1-M_1^2)k_1^2 - (1-M_2^2)k_2^2 + 2M\Omega(M_2k_2 - M_1k_1) - 2M_1M_2k_1k_2 \ . \tag{2.6}$$

In order to specify the value of $\gamma(k_1, k_2)$ branch cuts are required in both the k_1 and k_2 planes, and the nature of these cuts is dependent on whether the mean-flow component in the corresponding physical direction is subsonic or supersonic; since we are supposing throughout that the side edge is subsonic ($M_2 < 1$), the branch cuts in the k_2 plane will always be chosen to join the branch points in the upper and lower half-planes to infinity through the upper and lower half-planes respectively, with $\gamma(k_1, k_2)$ taking negative imaginary values as k_2 approaches infinity along the positive real axis. If the leading edge is also subsonic (i.e. $M_1 < 1$), then the branch cuts in the k_1 plane are specified in exactly the same way; for a supersonic leading edge ($M_1 > 1$), however, both branch points must be joined to infinity through the lower half of the k_1 plane.

We will apply the Wiener–Hopf technique with respect to the k_2 variable, but in order to achieve this it turns out that we must augment the normal-velocity boundary condition on the quarter-plane $\{x > 0, y > 0, z = 0\}$ by in addition specifying

$\partial\phi(x,y,z)/\partial z$ on the quadrant $\{x < 0, y > 0, z = 0\}$. We first note that, since the free stream is supersonic, the corner and side edge of the quarter-plane can have no upstream influence, and that therefore the scattered field on $\{x < 0, y > 0, z = 0\}$ must be exactly equal to that which would be observed for an infinite-span blade with no corner. Two cases now arise; first, if the mean-flow component normal to the leading edge is supersonic ($M_1 > 1$) then there can be no scattered field ahead of the leading edge, so that $\partial\phi(x,y,0)/\partial z \equiv 0$ in $x < 0, y > 0$; and second, when the leading edge has a subsonic velocity component the field in $x < 0, y > 0$ is non-zero, but as argued above can be determined by solving the simpler problem of scattering by an infinite-span blade. It can therefore be shown that (2.3) becomes

$$\frac{\partial \Phi^+}{\partial z}(k_1,k_2,0) - E(k_1,k_2) - \frac{V}{(k_1-\alpha_1)(k_2-\alpha_2)} = 0 \ , \tag{2.7}$$

where the plus superfix indicates that the y-integration has been taken over the semi-infinite range $y > 0$, and $E(k_1,k_2)$ is the as yet unspecified transform of $\partial\phi(x,y,0)/\partial z$ over $x < 0, y > 0$. As is usual in Wiener–Hopf problems, we shall require a multiplicative factorization of the kernel $\gamma(k_1,k_2)$ in the k_2 plane, in the form $\gamma(k_1,k_2) = \gamma^{+(2)}(k_1,k_2)\gamma^{-(2)}(k_1,k_2)$, with $\gamma^{\pm(2)}(k_1,k_2)$ analytic and non-zero in the upper and lower halves of the complex k_2 plane respectively (the superfix (2) is used to indicate that here the factorization of $\gamma(k_1,k_2)$ has been completed in the k_2 plane). By writing

$$\gamma^2(k_1,k_2) = -(1-M_2^2)(k_2-A)(k_2-B) \ , \tag{2.8}$$

where $A(k_1)$ and $B(k_1)$ are the branch points lying in the upper and lower halves of the k_2 plane, it becomes clear that we can take

$$\gamma^{+(2)}(k_1,k_2) = -\mathrm{i}(1-M_2^2)^{\frac{1}{2}}(k_2-B)^{\frac{1}{2}} \ , \tag{2.9}$$

with a similar expression for $\gamma^{-(2)}(k_1,k_2)$.

It now remains to determine expressions for the $E(k_1,k_2)$ of (2.7) for both M_1 less than unity and M_1 greater than unity, by considering the problem of scattering by an infinite-span blade. The analysis of the infinite-span problem can in fact be completed in a very straightforward manner by noting that the scattered potential must be of the form $\psi(x,z)\exp(-\mathrm{i}\alpha_2 y)$, and then by Fourier transforming with respect to x and application of the usual Wiener–Hopf arguments (this time with respect to the k_1 variable) we find that for $M_1 < 1$

$$\int_{-\infty}^{0} \frac{\partial \psi}{\partial z}(x,y,0)\exp(\mathrm{i}k_1 x)\mathrm{d}x = \frac{\mathrm{i}V\gamma^{-(1)}(k_1,\alpha_2)}{k_1-\alpha_1}\left[\frac{1}{\gamma^{-(1)}(k_1,\alpha_2)} - \frac{1}{\gamma^{-(1)}(\alpha_1,\alpha_2)}\right] \ , \tag{2.10}$$

where now $\gamma(k_1,k_2) = \gamma^{+(1)}(k_1,k_2)\gamma^{-(1)}(k_1,k_2)$, and $\gamma^{\pm(1)}(k_1,k_2)$ are analytic and non-zero in the upper and lower halves of the complex k_1 plane respectively; we emphasize that

$$\gamma^{\pm(1)}(k_1,k_2) \neq \gamma^{\pm(2)}(k_1,k_2) \ . \tag{2.11}$$

The expression for $E(k_1,k_2)$ with $M_1 < 1$ can now easily be derived from (2.10) – clearly $E(k_1,k_2) \equiv 0$ for $M_1 > 1$.

In the first instance we suppose that the leading edge has a subsonic velocity component ($M_1 < 1$), and substitute the expression for $E(k_1,k_2)$ derived above together with (2.5) into (2.7), to give

$$\tfrac{1}{2}\gamma^{+(2)}(k_1,k_2)[P(k_1,k_2,0)]^+_- - \frac{V(M\Omega - M_1k_1 + M_2k_2)\gamma^{-(1)}(k_1,\alpha_2)}{M(k_1-\alpha_1)(k_2-\alpha_2)\gamma^{-(1)}(\alpha_1,\alpha_2)\gamma^{-(2)}(k_1,\alpha_2)}$$
$$= \frac{V(M\Omega - M_1k_1 + M_2k_2)\gamma^{-(1)}(k_1,\alpha_2)}{M(k_1-\alpha_1)(k_2-\alpha_2)\gamma^{-(1)}(\alpha_1,\alpha_2)}\left\{\frac{1}{\gamma^{-(2)}(k_1,k_2)} - \frac{1}{\gamma^{-(2)}(k_1,\alpha_2)}\right\}$$
$$+\frac{(M\Omega - M_1k_1 + M_2k_2)}{M\gamma^{-(2)}(k_1,k_2)}\frac{\partial\Phi^-}{\partial z}(k_1,k_2,0) \ , \qquad (2.12)$$

where the minus superfix indicates that the y-integration has been completed over the range $y<0$ and $[P(k_1,k_2,0)]^+_-$ is the Fourier transform of the pressure jump across $z=0$. We now note that the left-hand side of (2.12) is analytic in the upper half of the k_2 plane, whilst the right-hand side is analytic in the lower half of the k_2 plane (as already noted, α_2 possesses a negative imaginary part and therefore lies in the lower half of the k_2 plane), and that (2.12) therefore defines a function, $F(k_1,k_2)$ say, which is analytic in the entire complex k_2 plane. The choice of $F(k_1,k_2)$ is now made by considering the behaviour of the scattered field in the limit $y \to +0$ (equivalent to the limit $k_2 \to \infty$ in the upper half-plane); we require the scattered field to be non-singular along $y=0$, and it turns out that in order to achieve this we must set $F(k_1,k_2)$ equal to a function of k_1 alone (i.e. $F(k_1)$), in such a way that $[P(k_1,k_2,0)]^+_- \propto k_2^{\frac{3}{2}}$ as $k_2 \to \infty$. We find that

$$F(k_1) = -\frac{VM_2\gamma^{-(1)}(k_1,\alpha_2)}{M(k_1-\alpha_1)\gamma^{-(1)}(\alpha_1,\alpha_2)\gamma^{-(2)}(k_1,\alpha_2)} \ . \qquad (2.13)$$

This specification is entirely equivalent to the imposition of a Kutta condition (see Crighton 1985) along the side edge $y=0$, and corresponds physically to the fact that vorticity will be shed from the side edge of the quarter-plane so as to ensure that the scattered field remains non-singular at the side edge. Clearly, this vortex shedding can only occur for a non-zero sweep angle (i.e. when the mean flow possess a non-zero component normal to and away from the side edge), and in the special case of $M_2=0$, when the mean flow cannot convect away vorticity shed from the side edge, the required choice of $F(k_1)$ is identically zero. The solution of the problem can now be completed, and we find that the scattered pressure for $M_1<1$ is given by

$$p(x,y,z) = -\frac{\operatorname{sgn} z}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}k_1\mathrm{d}k_2 \frac{\exp(-\mathrm{i}k_1x - \mathrm{i}k_2y - i\gamma|z|)}{(k_1-\alpha_1)(k_2-\alpha_2)}$$
$$\times\frac{V\gamma^{-(1)}(k_1,\alpha_2)(M_1k_1 - M\Omega - M_2\alpha_2)}{M\gamma^{+(2)}(k_1,k_2)\gamma^{-(1)}(\alpha_1,\alpha_2)\gamma^{-(2)}(k_1,\alpha_2)} \ , \qquad (2.14)$$

with the integration contour suitably indented above and below the various branch points and poles. We note that the analysis for the case of a supersonic leading edge ($M_1>1$) can be completed in much the same way, and it turns out that the expression for the scattered pressure is exactly equal to that derived above in the subsonic case, but with the factors $\gamma^{-(1)}(\alpha_1,\alpha_2)$ and $\gamma^{-(1)}(k_1,\alpha_2)$ set equal to unity.

Our concern in this paper is with the unsteady pressure on the blade surface, but it has not proved possible to find a closed-form expression for the detailed lift distribution $[p(x,y,0)]^+_-$ from (2.14), and we proceed instead by deriving an expression for the lift per unit span, $\mathscr{L}(y)$, defined by

$$\mathscr{L}(y) = \int_0^{\infty}[p(x,y,0)]^+_-\mathrm{d}x \ ; \qquad (2.15)$$

as argued in the introduction, $\mathscr{L}(y)$ will be in good agreement with the lift integrated over the actual finite chord, provided that $\mathscr{L}(y)$ converges. We extend the x-integration in (2.15) to $-\infty$, substitute the expression for the pressure jump in terms of $[P(k_1,k_2,0)]_-^+$ derived above and complete the x-integral as $2\pi\delta(k_1)$. The remaining k_2 integral can then be completed by deforming the contour round the branch cut in the lower half-plane; the branch-cut contribution is evaluated using a result from §7.1.3 of Abramowitz & Stegun (1968), and adding in the contribution from the pole at $k_2 = \alpha_2$. It turns out that the lift per unit span on our semi-infinite span blade for $M_1 < 1$ is

$$\mathscr{L}(y) = \frac{2\mathrm{i}V(M\Omega + M_2\alpha_2)\gamma^{-(1)}(0,\alpha_2)\exp(-\mathrm{i}\alpha_2 y)}{M\alpha_1\gamma^{-(1)}(\alpha_1,\alpha_2)\gamma(0,\alpha_2)} \times \mathrm{erf}[\gamma^{+(2)}(0,\alpha_2)\exp(\mathrm{i}\pi/4)y^{\frac{1}{2}}/(1-M_2^2)^{\frac{1}{2}}] \; . \tag{2.16}$$

For the $M_1 > 1$ result we again simply set $\gamma^{-(1)}$ equal to unity in (2.16). The rather complicated algebraic expressions for the various factors of γ appearing in (2.16) are given in the Appendix. In addition, an expression for the *acoustically weighted* lift per unit span (which is used directly in noise-prediction codes, and which is defined in exactly the same way as $\mathscr{L}(y)$ apart from an extra factor $\exp(\mathrm{i}Kx)$ to account for retarded-time differences along the chord, where the wavenumber K depends on the various operating parameters), could be derived by a trivial modification to the above (Peake 1992).

It is easy to show that the lift per unit span on an *infinite-span* blade, denoted $\mathscr{L}_{inf}(y)$, would be the same as the expression given in (2.16) but without the error-function factor, i.e. for $M_1 < 1$

$$\mathscr{L}_{inf}(y) = \frac{2\mathrm{i}V(M\Omega + M_2\alpha_2)\gamma^{-(1)}(0,\alpha_2)\exp(-\mathrm{i}\alpha_2 y)}{M\alpha_1\gamma^{-(1)}(\alpha_1,\alpha_2)\gamma(0,\alpha_2)} \; . \tag{2.17}$$

Equation (2.16) is the principal mathematical result of this paper, and will be used in the next section to study the dependence of the unsteady lift on the various flow parameters, but before doing that we will consider the possibility of $\mathscr{L}(y)$ and $\mathscr{L}_{inf}(y)$ becoming infinite for certain critical values of the flow parameters, as follows.

2.3. *Singularities in the lift per unit span*

First, in the *infinite-span* case it is clear that our expression for $\mathscr{L}_{inf}(y)$ will become infinite when $\gamma^{+(1)}(0,\alpha_2)$ vanishes (the factor $\gamma^{-(1)}(\alpha_1,\alpha_2)$ cannot vanish for any choice of α_1 and α_2, since α_1 lies in the lower half-plane and $\gamma^{-(1)}(k_1,k_2)$ was defined so as to be non-zero in the lower half of the k_1 plane), and in fact $\gamma^{+(1)}(0,\alpha_2)$ has the two zeros

$$\alpha_2 = \frac{M\Omega}{1-M_2} \quad \text{and} \quad \alpha_2 = \frac{-M\Omega}{1+M_2} \; . \tag{2.18}$$

There are therefore two critical values of α_2 (the component of the gust phase-front trace speed along the leading edge) for which $\mathscr{L}_{inf}(y)$ becomes infinite, and it is easy to see that they correspond to the values of the sonic velocity along the leading edge in the positive y-direction and in the negative y-direction respectively. The detailed lift distribution generated by a gust striking the infinite-span blade, $[p_{2d}(x,y,0)]_-^+$, can be shown to be proportional to $x^{-\frac{1}{2}}\exp(\mathrm{i}\chi x)$ for fixed y, where the chordwise wavenumber χ depends on the incident-gust wavenumbers; when α_2 takes either of the values (2.18) we find that $\chi = 0$, so that the chordwise integral of $[p_{2d}(x,y,0)]_-^+$ over the infinite chord of the quarter-plane does not converge. We again emphasize that in these two critical cases the gust trace speed along the leading edge is exactly sonic, so that no spanwise separation can occur between the scattered sound field and the incident gust, causing the chordwise wavelength of the scattered field to become infinite and leading to the infinities in $\mathscr{L}_{inf}(y)$.

Second, in the *semi-infinite span* case it can be shown from (2.16) that $\mathscr{L}(y)$ also becomes infinite when $\alpha_2 = -M\Omega/(1+M_2)$, corresponding to the gust trace speed along the leading edge being exactly equal to the sonic velocity in the *negative* y-direction. However, it turns out that $\mathscr{L}(y)$ is *finite* when $\alpha_2 = M\Omega/(1-M_2)$, which is when the gust trace speed along the leading edge coincides with the sonic velocity in the *positive* y-direction. Mathematically this is because the critical value $\alpha_2 = M\Omega/(1-M_2)$ corresponds to the single zero of $\gamma^{+(2)}(0,\alpha_2)$; with (2.16) in mind we define

$$Q(\alpha_2) \equiv \left| \frac{\operatorname{erf}[\gamma^{+(2)}(0,\alpha_2)\exp(i\pi/4)y^{\frac{1}{2}}/(1-M_2^2)^{\frac{1}{2}}]}{\gamma_2^{+(2)}(0,\alpha_2)} \right| , \tag{2.19}$$

so that

$$Q(\alpha_2) \sim \frac{2y^{\frac{1}{2}}}{(1-M_2^2)^{\frac{1}{2}}\pi^{\frac{1}{2}}} \quad \text{as} \quad \alpha_2 \to \frac{M\Omega}{1-M_2} , \tag{2.20}$$

and hence it follows that the zero of the error function has the effect of cancelling the zero in the denominator in (2.16), leading to a *finite* value of $\mathscr{L}(y)$ for this critical value of α_2. Moreover, it is easy to show that the function $Q(\alpha_2)$ attains its maximum value for $\alpha_2 = M\Omega/(1-M_2)$, and since the argument of the error function in $Q(\alpha_2)$ is proportional to $\Omega^{\frac{1}{2}}$ and we are concerned here with moderate and large values of Ω, it follows that $Q(\alpha_2)$ is a relatively sharply peaked function near $\alpha_2 = M\Omega/(1-M_2)$ and will therefore tend to dominate the behaviour of $\mathscr{L}(y)$, provided that y is not exceedingly small. It can therefore be seen that as α_2 is varied the critical value $\alpha_2 = M\Omega/(1-M_2)$ will yield a large (local maximum) value of $|\mathscr{L}(y)|$ all along the span, apart from in an effectively unimportant region for which $y \ll 1$.

We have seen that for the infinite-span blade the chordwise oscillations in lift distribution will vanish (i.e. infinite chordwise wavelength), leading to the singularities in $\mathscr{L}_{inf}(y)$, when the spanwise gust trace speed is equal to either of the sonic velocities along the leading edge, but that for the quarter-plane this only arises when the spanwise trace speed is equal to the sonic velocity in the negative y-direction – $\mathscr{L}(y)$ is finite when the spanwise trace speed coincides with the sonic velocity in the positive y-direction. We can therefore conclude that the infinite chordwise wavelength, and consequent divergence of $\mathscr{L}_{inf}(y)$, observed when α_2 is equal to the values (2.18) is associated with the unrealistic *infinite* extent of the blade in the negative and positive spanwise directions respectively. It follows that if one were to consider a finite-span blade, then no singularity in $\mathscr{L}(y)$ would be present, and that $\mathscr{L}(y)$ should be in good agreement with the lift integrated over the actual finite chord for all α_2, including the two critical values (2.18). The arguments following (2.20) suggest that the maximum (finite) amplitude of the unsteady response of a finite-span blade would occur when α_2 is equal to either of these two critical wavenumbers.

Finally, we note that as well as becoming infinite at one or both of the critical values of α_2 given in (2.18), $\mathscr{L}(y)$ and $\mathscr{L}_{inf}(y)$ are also infinite in the special case $\alpha_1 = 0$. This additional singularity corresponds to the gust wavenumber along the chord becoming identically zero, and could presumably be resolved by including the trailing edge in the analysis. This case will not be considered further; in the examples which follow α_1 is always large and positive

3. Results and discussion

We now proceed to consider the behaviour of $\mathscr{L}(y)$ along the blade span and for various typical parameter values. It can easily be seen from (2.16) that the lift approaches zero close to the side edge (in fact $\mathscr{L}(y) \propto y^{\frac{1}{2}}$ as $y \to +0$), and this corresponds to the fact that the pressure difference across the quarter-plane induces an unsteady flow around the side edge, from the pressure surface to the suction surface, which acts so as to reduce the pressure difference to zero as the side edge is approached. In addition, the effect of the presence of the side edge tends to zero very far from the side edge, since the error-function factor in (2.16) approaches unity as $y \to \infty$. In figure 2 we plot $|\mathscr{L}(y)/V|$ against spanwise coordinate, y, for $M = 1.2$ and $\Omega = k_p = 5$, in the two cases of $\Lambda = 0$ (unswept, supersonic leading edge) and $\Lambda = \pi/4$ (swept, subsonic leading edge) – in both these cases α_2 is very far from the two critical values identified in the previous section. Both curves exhibit the behaviour described above as $y \to 0$ and $y \to \infty$, but the lengthscale over which $\mathscr{L}(y)$ approaches its infinite-span limit (and hence the portion of the blade span over which the tip effects predominate) differs widely in the two cases. For $\Lambda = \pi/4$ ($M_1 = M_2 = 0.85$) the spanwise distance taken for the lift to adjust from the value zero at the side edge to a value relatively close to the corresponding infinite-span result is of the order of only several gust wavelengths, whereas the transition takes place over a much longer lengthscale for the $\Lambda = 0$ case (indeed, for the unswept blade with these parameters $|\mathscr{L}(y)/V|$ will only approach within 10% of $|\mathscr{L}_{inf}(y)/V|$ for $y \geqslant 30$ – i.e. beyond 30 chord lengths from the tip). This behaviour can be understood by writing the modulus of the argument of the error function as $(y/l)^{\frac{1}{2}}$, so that the spanwise lengthscale, l, over which $\mathscr{L}(y)$ approaches its infinite-span limit satisfies

$$l \propto \left(\frac{M\Omega}{1-M_2} - \alpha_2\right)^{-1} . \qquad (3.1)$$

If we consider some given incident gust (i.e. fix Ω and k_{p}), and increase the sweep angle from zero, then the magnitude of the first term in (3.1) increases monotonically and becomes large, whilst the value of α_2 remains bounded (the precise behaviour of α_2 as Λ increases depends on the relative values of the gust wavenumbers Ω and k_p). We can therefore conclude that, provided Λ (and hence M_2) are not too small, the first term in (3.1) will tend to dominate the second term, and it follows that l will be much smaller for a swept blade with a subsonic leading edge than for an unswept blade encountering the same gust. *The portion of the span over which the tip region has a significant influence on the lift distribution is therefore seen to be significantly reduced by sweep.* This conclusion is supported in figure 3, where we plot the tip correction coefficient, C, defined by

$$C \equiv \left|\frac{\mathscr{L}(y) - \mathscr{L}_{inf}(y)}{\mathscr{L}_{\mathrm{inf}}(y)}\right| , \qquad (3.2)$$

which measures the magnitude of the effect of the blade tip; the portion of the span over which C differs significantly from zero is seen to decrease markedly as the sweep angle Λ increases. In addition, we can see from (3.1) that for $k_p = O(\Omega)$ we have $l \propto \Omega^{-1}$, so that the spanwise extent of the region of influence of the tip decreases with increasing gust frequency. Figure 2 indicates that sufficiently far from the tip the discrepancy between $\mathscr{L}(y)$ and $\mathscr{L}_{inf}(y)$ takes the form of a slowly decaying spanwise oscillation, and by considering the second term in the asymptotic expansion of the error function in (2.16) as $y \to \infty$, it follows that, sufficiently far from the tip, the

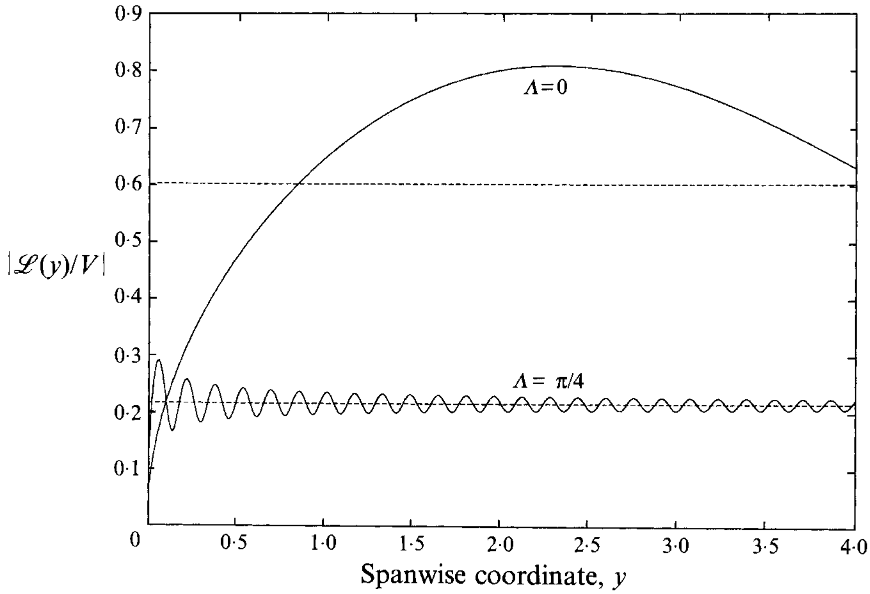

FIGURE 2. Plots of the modulus of the normalized lift per unit span for the quarter-plane, $|\mathscr{L}(y)/V|$ (solid lines), and for the corresponding infinite-span blade, $|\mathscr{L}_{inf}(y)/V|$ (dashed lines), against chordwise coordinate y, with $M = 1.2$, $\Omega = k_p = 5$ and with $\Lambda = 0$ (unswept, supersonic leading edge) and $\Lambda = \pi/4$ (swept, subsonic leading edge).

effect of the unsteady flow near the tip on $\mathscr{L}(y)$ is to generate a wave of wavelength l and amplitude proportional to $y^{-\frac{1}{2}}$ propagating away from the tip.

In figure 4(*a–c*) we select a single spanwise station (here $y = 1$, although other choices of y would give plots exhibiting qualitatively the same behaviour) and consider the variation of $|\mathscr{L}(1)/V|$ and $|\mathscr{L}_{inf}(1)/V|$ as the sweep angle, Λ, is increased from zero to a value just less than $\sin^{-1} 1/M_2$ (so that the mean flow component perpendicular to the side edge never becomes supersonic) with M fixed, for three different sets of values of the gust wavenumbers Ω and k_p. In figure 4(*a*) we have $\Omega = k_p = 5$, and for this choice of parameters neither of the critical values of α_2 given in (2.18) lie in our Λ-range. It can be seen that here the infinite-span lift $\mathscr{L}_{inf}(1)$ decreases from its maximum value at zero sweep to a much lower value once the leading edge has become subsonic, and that, in agreement with our earlier arguments, the discrepancy between $\mathscr{L}(1)$ and $\mathscr{L}_{inf}(1)$, and hence the level of tip effects, is largest for the smaller sweep angles. In figure 4(*b*) we have $\Omega = 5$ and $k_p = 0$, and here the critical value $\alpha_2 = -M\Omega/(1+M_2)$, for which both $\mathscr{L}(1)$ and $\mathscr{L}_{inf}(1)$ become infinite, arises in our sweep range; for this choice of incident-gust wavenumbers the unsteady response of the blade would increase as the sweep angle is increased from zero up to the point where the critical value of α_2 is reached, and would then tend to decrease for larger sweep angles. Finally, in figure 4(*c*) we choose $\Omega = 5$ and $k_p = 10$, and here the other critical value $\alpha_2 = M\Omega/(1-M_2)$, for which $\mathscr{L}_{2d}(1)$ is singular but $\mathscr{L}(1)$ is non-singular, arises in our Λ-range. In particular, we note that $|\mathscr{L}(1)|$ takes a relatively large (maximum) value at this critical point, in agreement with our earlier arguments, before being reduced as the sweep angle is increased further. It is therefore clear from our sequence of figures (which are representative of the behaviour for any spanwise coordinate) that the effect of blade sweep can be to either decrease or increase $|\mathscr{L}(y)/V|$, in a way that depends on the values of the gust wavenumber and

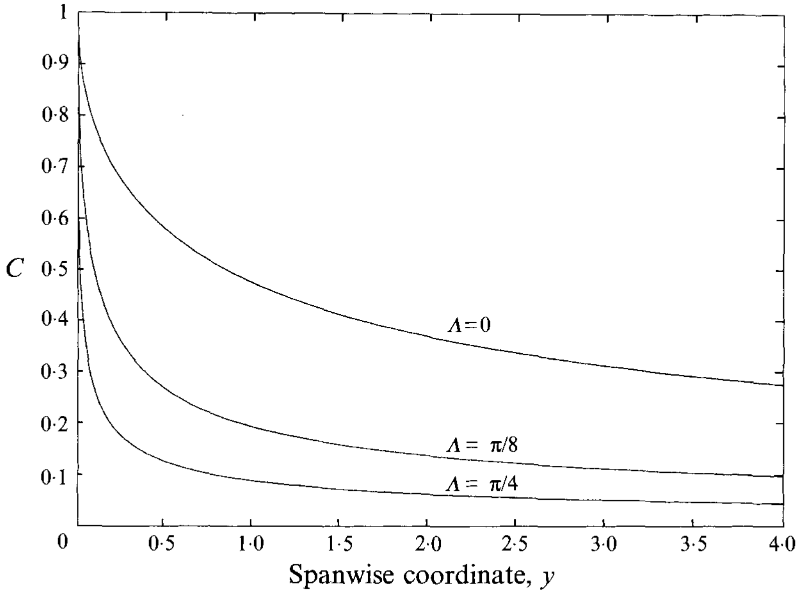

FIGURE 3. Plots of the modulus of the tip correction coefficient, C, against spanwise coordinate y, for various values of Λ and with other conditions as in figure 2 (the case $\Lambda = \pi/8$ corresponds to a swept, supersonic leading edge).

on the choice of sweep angle. It is worth noting that the maximum in the value of $|\mathscr{L}(y)/V|$ coincident with the critical value $\alpha_2 = M\Omega/(1 - M_2)$, as observed in figure 4(*c*) for $y = 1$, was found to occur for almost all y (certainly for all $y > 0.2$), in full agreement with the arguments made in §2.3. The increasing frequency of the oscillations of our $\mathscr{L}(y)$-curves in figure 4(*a–c*) with increasing Λ can be understood by considering the behaviour of the argument of the error-function factor in (2.16), since the factor $\gamma^{+(2)}(0, \alpha_2)$ tends to increase with increasing sweep. It can also be seen from these figures that our expressions for $\mathscr{L}(y)$ with $M_1 > 1$ and $M_1 < 1$ join smoothly at $M_1 = 1$.

4. Concluding remarks

By considering the behaviour of our expression for $\mathscr{L}(y)$, we have demonstrated how the region of influence of the tip along the blade span will tend to be reduced by increasing either the sweep angle or the gust frequency. However, it is clear from figure 3 that, even for the highly swept subsonic leading-edge case, there is a significant discrepancy between the actual unsteady lift on the quarter-plane and what would be predicted using infinite-span theory over a number of chord lengths from the tip. Since modern fan and propeller blades tend to have long chords, this suggests that the use of infinite-span response theory is unjustified over much of the span, and that application of the sort of formulae developed in this paper is required not merely near the tip, but in fact along most of the blade.

For an infinite-span blade there are two critical values of the spanwise gust wavenumber for which the gust moves sonically along the leading edge, and in each of these cases the chordwise oscillation in the scattered field vanishes, leading to an infinity in our expression for $\mathscr{L}(y)$. We have demonstrated that one of these singularities is removed when the infinite-span blade is replaced by a quarter-plane,

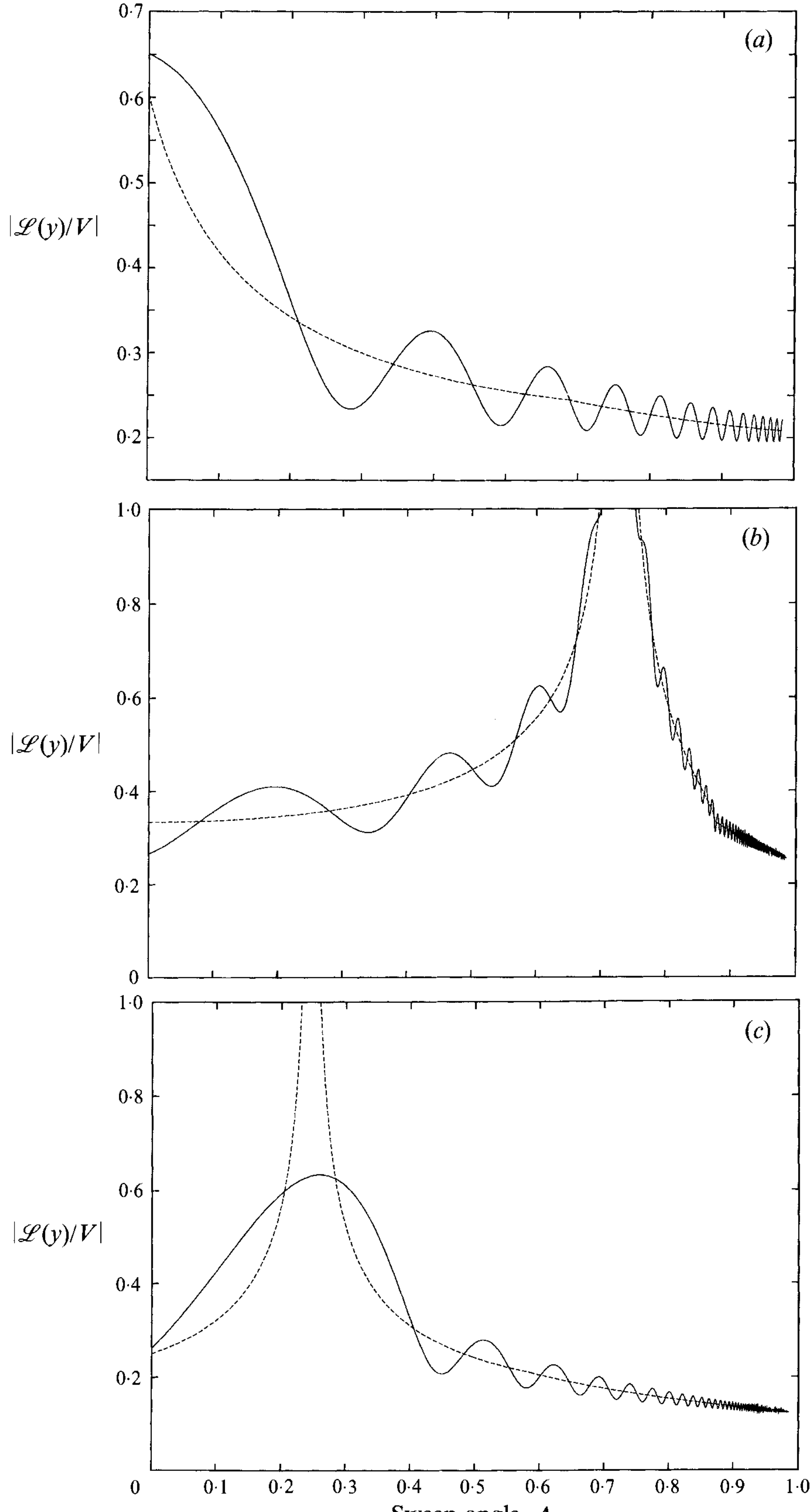

FIGURE 4. Plots of $|\mathscr{L}(y)/V|$ (solid lines) and $|\mathscr{L}_{inf}(y)/V|$ (dashed lines) against sweep angle Λ, with $M = 1.2$, $y = 1$ and with (a) $\Omega = k_p = 5$; (b) $\Omega = 5$, $k_p = 0$; (c) $\Omega = 5$, $k_p = 10$.

and that as the sweep angle is varied the lift per unit span will take a local maximum, but finite, value at this critical condition. It follows that for a finite-span blade the lift per unit span will take its largest value when the gust wavenumber along the leading edge equals either of the two critical values given in (2.18). This should be of practical importance, since it suggests that the unsteady response of the entire blade, and hence the noise generated, could be particularly large for the certain gust–sweep combinations identified in this paper.

Finally, we emphasize that a closed-form expression for the acoustically weighted lift can be found via a trivial modification of our formula for $\mathscr{L}(y)$, which could then be included in practical propeller and fan noise prediction schemes to account for the tip effects of swept blades.

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Appendix

In this Appendix we present algebraic expressions for the various factors of $\gamma(k_1, k_2)$ which appear in our expressions for the lift – equations (2.16) and (2.17). In the case $k_1 = 0$ we have that

$$\gamma^2(0, k_2) = M^2\Omega^2 + 2MM_2\Omega k_2 - (1 - M_2^2)k_2^2 \ , \tag{A 1}$$

so that the Wiener–Hopf factorization in the k_2 plane can be completed in the relatively compact form

$$\left.\begin{aligned} \gamma^{+(2)}(0, k_2) &= -\mathrm{i}(1 - M_2^2)^{\frac{1}{2}} \left(k_2 - \tfrac{M\Omega}{1-M_2}\right)^{\frac{1}{2}}, \\ \gamma^{-(2)}(0, k_2) &= \left(k_2 + \tfrac{M\Omega}{1+M_2}\right)^{\frac{1}{2}} \ , \end{aligned}\right\} \tag{A 2}$$

with the branch cuts for the square roots lying in the lower and upper halves of the k_2 plane respectively.

In addition, we require the quantity $\gamma^{-(1)}(k_1, \alpha_2)$ for real k_1, and in order to do this in the case $M_1 < 1$ we need to locate the branch point, K, of $\gamma(k_1, \alpha_2)$ lying in the upper half of the k_1 plane. Defining

$$\left.\begin{aligned} C &\equiv M_1M_2\alpha_2 + MM_1\Omega, \\ D &\equiv \alpha_2^2(M^2 - 1) + 2\alpha_2MM_2\Omega + M^2\Omega^2 \ , \end{aligned}\right\} \tag{A 3}$$

we find that

$$\left.\begin{aligned} K &= \frac{-C - D^{\frac{1}{2}}}{1 - M_1^2} \quad \text{for } D > 0, \\ K &= \frac{-C + \mathrm{i}|D|^{\frac{1}{2}}}{1 - M_1^2} \quad \text{for } D < 0 \ . \end{aligned}\right\} \tag{A 4}$$

It then follows that

$$\gamma^{-(1)}(k_1, \alpha_2) = (k_1 - K)^{\frac{1}{2}} \ , \tag{A 5}$$

with the branch cut for the square root lying in the upper half-plane. As already stated, the factor $\gamma^{-(1)}(k_1, k_2)$ is set equal to unity for $M_1 > 1$.

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